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EXACT SOLUTION FOR FREEZING OF HUMID POROUS HALF-SPACE*

M. D. MIKHAILOV
Applied Mathematics Centre, Sofia C, P.B.384, Bulgaria

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Abstract—New Stephan-like problem is defined and exact solutions for temperature and moisture distributions as well as the position of the moving freezing front in a humid porous half-space are obtained. The analytical solution is programmed in BASIC and some illustrative examples are plotted.

NOMENCLATURE

A_1, A_2, A_3,	constants of integration;
a_1, a_2,	thermal diffusivity;
a_m,	moisture diffusivity;
B_1, B_2, B_3,	constants of integration;
c_2,	specific heat capacity;
Fo, Fo_x,	Fourier number, $a_2\tau/l^2$ and $a_2\tau/x^2$ respectively;
k_1, k_2,	thermal conductivity;
Ko,	Kossovitch number, $ru_0/(c_2\Delta t)$;
l,	characteristic length;
Lu,	Luikov number, a_m/a_2;
Pn,	Posnov number, $\delta\Delta t/u_0$;
r,	latent heat of freezing;
$s(\tau)$,	position of freezing front;
$S(Fo)$,	nondimensional position of freezing front, s/l;
$t_1(x,\tau), t_2(x,\tau)$,	temperature;
$T_1(X,Fo), T_2(X,Fo)$,	nondimensional temperature, $[t_1(x,\tau)-t_0]/\Delta t$ and $[t_2(x,\tau)-t_0]/\Delta t$ respectively;
$u(x,\tau)$,	moisture content of a body;
x,	length coordinate;
X,	nondimensional length, x/l;
$Z(X,Fo)$,	potential defined by equation (17);
erf( ),	error function;
erfc( ),	complimentary error function.

Greek symbols

δ,	thermal gradient coefficient;
$\theta(X,Fo)$,	nondimensional moisture potential, $[u_0-u(x,\tau)]/u_0$;
λ,	nondimensional constant, $S/[2(Fo)^{\frac{1}{2}}]$;
τ,	time.

Subscripts

1,	first region, $0 < X < S$, $1/(4\lambda^2) < Fo_x < \infty$;
2,	second region, $S < X < \infty$, $0 < Fo_x < 1/(4\lambda^2)$;
s,	at surface $X = 0$;
v,	at freezing front $X = S$;
21, 12,	ratio of properties of region 2 to 1 and 1 to 2 respectively.

1. INTRODUCTION

SINCE 1963, in Laboratoire d'Aérothermique de Meudon, fundamental research has been performed on the freezing of porous media [1, 2]. In the papers cited it is shown experimentally that the freezing of soil is accompanied with a set of interconnected phenomena among which an important role is played by the migration of moisture towards the freezing front.

The well-known mathematical model, formulated by Stephan [3] does not account for the movement of moisture. But in connection with the drying theory new more complicated Stephan-like problems were studied in [4].

In the present study a similar model is defined corresponding to the freezing of humid porous half-space and the exact solution is obtained for temperature and moisture distributions as well as for the position of the moving freezing front. The analytical solution is coded in BASIC for minicomputer WANG 2200 and some illustrative examples are plotted.

2. STATEMENT OF THE PROBLEM

Let us consider the flow of heat and moisture through a porous half-space during freezing. The position of phase change front at time τ is given by $x = s(\tau)$. It divides the porous body into two regions.

In the freezing region, $0 < x < s(\tau)$, there is no moisture movement and the temperature distribution is described by

$$\frac{\partial t_1(x,\tau)}{\partial \tau} = a_1 \frac{\partial^2 t_1(x,\tau)}{\partial x^2}. \tag{1}$$

The region $s(\tau) < x < \infty$ is humid capillary porous body in which there are coupled heat and moisture flows. The process is described by the well known Luikov's system [5], for the case $\varepsilon = 0$:

$$\frac{\partial t_2(x,\tau)}{\partial \tau} = a_2 \frac{\partial^2 t_2(x,\tau)}{\partial x^2} \tag{2}$$

$$\frac{\partial u(x,\tau)}{\partial \tau} = a_m \frac{\partial^2 u(x,\tau)}{\partial x^2} + a_m\delta \frac{\partial^2 t_2(x,\tau)}{\partial x^2}. \tag{3}$$

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The initial distributions of temperature and moisture are uniform

$$t_2(x,0) = t_2(\infty,\tau) = t_0, \quad u(x,0) = u(\infty,\tau) = u_0. \quad (4)$$

It is also assumed that on the surface of the half-space the temperature is constant but differing from the initial one

$$t_1(0,\tau) = t_s, \quad (5)$$

where $t_s < t_v$.

On the freezing front, exists an equality between the temperatures

$$t_1(s,\tau) = t_2(s,\tau) = t_v. \quad (6)$$

Heat and moisture balance at the freezing front yields

$$k_1\frac{\partial t_1(s,\tau)}{\partial x} - k_2\frac{\partial t_2(s,\tau)}{\partial x} = u(s,\tau)\rho_2 r\frac{\mathrm{d}s(\tau)}{\mathrm{d}\tau} \quad (7)$$

$$\frac{\partial u(s,\tau)}{\partial x} + \delta\frac{\partial t_2(s,\tau)}{\partial x} = 0. \quad (8)$$

The set of equations (1)–(3) can be put in a non-dimensional form as follows

$$\frac{\partial T_1(X,Fo)}{\partial Fo} = a_{12}\frac{\partial^2 T_1(X,Fo)}{\partial X^2} \quad (9)$$

$$\frac{\partial T_2(X,Fo)}{\partial Fo} = \frac{\partial^2 T_2(X,Fo)}{\partial X^2} \quad (10)$$

$$\frac{\partial\theta(X,Fo)}{\partial Fo} = Lu\frac{\partial^2\theta(X,Fo)}{\partial X^2} - Lu\,Pn\frac{\partial^2 T_2(X,Fo)}{\partial X^2}. \quad (11)$$

The initial and boundary conditions are

$$T_2(X,0) = T_2(\infty,Fo) = 0, \quad \theta(X,0) = \theta(\infty,Fo) = 0 \quad (12)$$

$$T_1(0,Fo) = T_s. \quad (13)$$

The interface conditions are

$$T_1(S,Fo) = T_2(S,Fo) = T_v \quad (14)$$

$$\frac{\partial\theta(S,Fo)}{\partial X} - Pn\frac{\partial T_2(S,Fo)}{\partial X} = 0 \quad (15)$$

$$k_{12}\frac{\partial T_1(S,Fo)}{\partial X} - \frac{\partial T_2(S,Fo)}{\partial X} = Ko[1-\theta(S,Fo)]\frac{\mathrm{d}S(Fo)}{\mathrm{d}Fo}. \quad (16)$$

3. SOLUTION OF THE PROBLEM

For convenience in the derivation of the solution, we introduce now a new variable:

$$Z(X,Fo) = T_2(X,Fo) + C\theta(X,Fo). \quad (17)$$

After multiplying equation (11) with a constant C and adding the result to equation (10), one obtains

$$\frac{\partial}{\partial Fo}\{T_2(X,Fo)+C\theta(X,Fo)\} = (1-C\,Lu\,Pn) \times\frac{\partial^2}{X^2}\left\{T_2(X,Fo)+\frac{C\,Lu}{1-C\,Lu\,Pn}\theta(X,Fo)\right\}. \quad (18)$$

In order that equation (18) be of the pure heat conduction type, one has to assume that

$$C = \frac{C\,Lu}{1-C\,Lu\,Pn} \quad (19)$$

or

$$C = \frac{1-Lu}{Lu\,Pn}. \quad (20)$$

Equation (18) becomes

$$\frac{\partial Z(X,Fo)}{\partial Fo} = Lu\frac{\partial^2 Z(X,Fo)}{\partial X^2}. \quad (21)$$

Then the pure heat conduction type differential equations, like equations (9), (10) and (21) have the following solutions:

$$T_1(X,Fo) = A_1 + B_1\,\mathrm{erf}\left(\frac{X}{2(a_{12}Fo)^{\frac{1}{2}}}\right) \quad (22)$$

$$T_2(X,Fo) = A_2 + B_2\,\mathrm{erf}\left(\frac{X}{2(Fo)^{\frac{1}{2}}}\right) \quad (23)$$

$$Z(X,Fo) = A_3 + B_3\,\mathrm{erf}\left(\frac{X}{2(Lu\,Fo)^{\frac{1}{2}}}\right). \quad (24)$$

Substitution of equations (20), (23) and (24) in (17) yields

$$\theta(X,Fo) = \frac{Lu\,Pn}{1-Lu}\left\{A_3 + B_3\,\mathrm{erf}\left(\frac{X}{2(Lu\,Fo)^{\frac{1}{2}}}\right) - A_2 - B_2\,\mathrm{erf}\left(\frac{X}{2(Fo)^{\frac{1}{2}}}\right)\right\}. \quad (25)$$

The constants A_1, A_2, A_3, B_1, B_2 and B_3 have to be chosen so that they satisfy the initial and boundary conditions (12)–(16). For the case under consideration, this is possible and consequently the problem has an exact analytical solution.

From the initial condition (12), one obtains a system of two algebraic equations and its solution yields

$$B_2 = -A_2, \quad B_3 = -A_3. \quad (26)$$

The boundary condition (13) leads to

$$A_1 = T_s. \quad (27)$$

From the condition (14), one obtains

$$T_s + B_1\,\mathrm{erf}[\lambda/(a_{12})^{\frac{1}{2}}] = A_2\,\mathrm{erfc}\,\lambda = T_v, \quad (28)$$

where

$$\lambda = \frac{S}{2(Fo)^{\frac{1}{2}}}.$$

It follows from equations (15) and (16) that:

$$(Lu)^{\frac{1}{2}}B_3\exp(-\lambda^2/Lu) - B_2\exp(-\lambda^2) = 0 \quad (29)$$

and

$$B_1\frac{k_{12}}{(a_{12})^{\frac{1}{2}}}\exp(-\lambda^2/a_{12}) - B_2\exp(-\lambda^2) = \pi^{\frac{1}{2}}\lambda\,Ko\left\{1-\frac{Lu\,Pn}{1-Lu}[A_3\,\mathrm{erfc}[\lambda/(Lu)^{\frac{1}{2}}] - A_2\,\mathrm{erfc}(\lambda)]\right\}. \quad (30)$$

After determining the constants A_1, A_2, A_3, B_1, B_2 and B_3 from equations (26) to (29), the solutions (22),

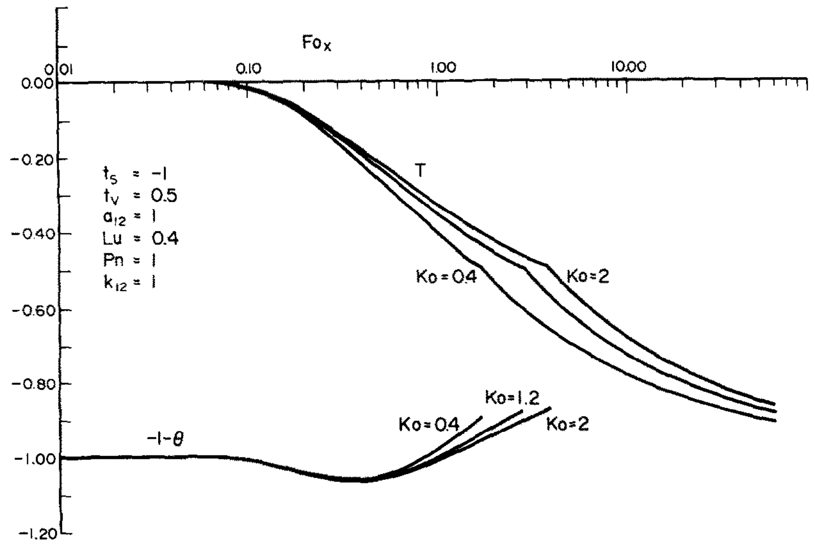

FIG. 1. Influence of *Ko* on non-dimensional temperature and moisture.

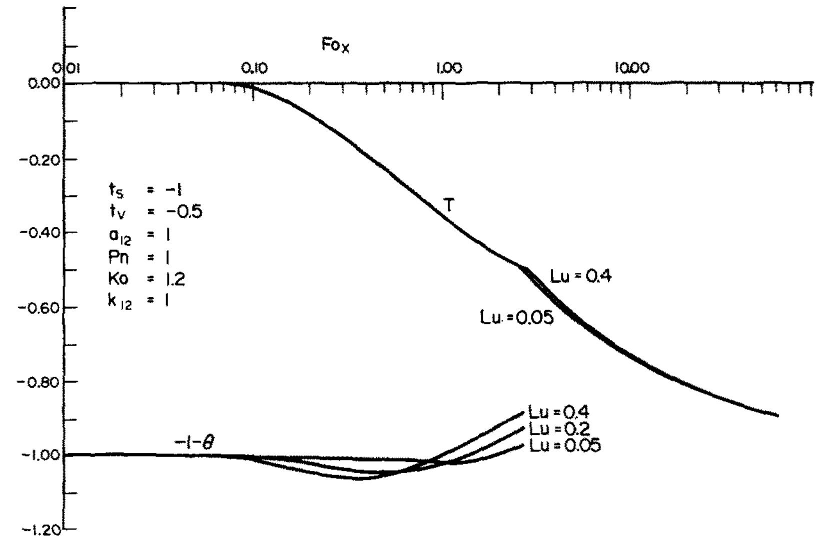

FIG. 2. Influence of *Lu* on non-dimensional temperature and moisture.

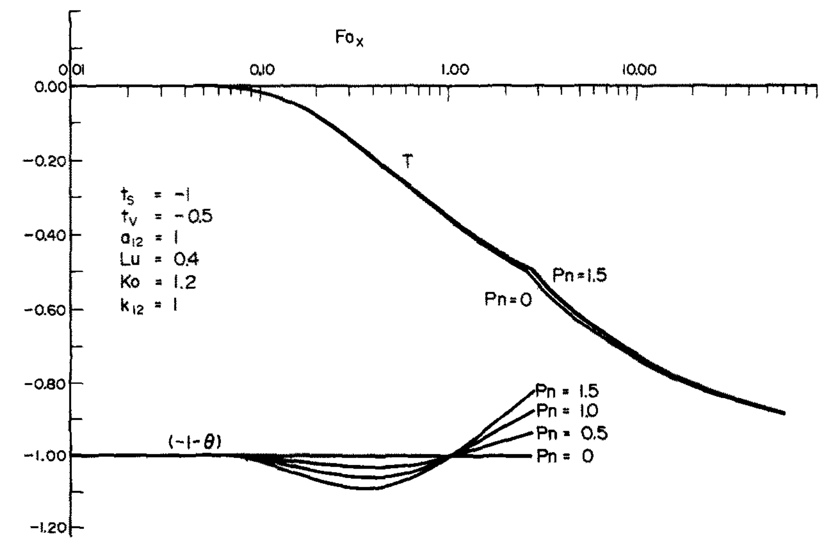

FIG. 3. Influence of *Pn* on non-dimensional temperature and moisture.

(23) and (25) can be written as

$$T_1(Fo_x) = T_s + \frac{T_v - T_s}{\operatorname{erf}\left[\frac{\lambda}{(a_{12})^{\frac{1}{2}}}\right]} \operatorname{erf}\left[\frac{1}{2(a_{12}Fo_x)^{\frac{1}{2}}}\right], \quad 1/(4\lambda^2) < Fo_x < \infty \quad (31)$$

$$T_2(Fo_x) = \frac{T_v}{\operatorname{erfc}(\lambda)} \operatorname{erfc}\left(\frac{1}{2(Fo_x)^{\frac{1}{2}}}\right), \quad 0 < Fo_x < 1/(4\lambda^2) \quad (32)$$

$$\theta(Fo_x) = T_v \frac{Lu\,Pn}{1-Lu}\left\{\frac{1}{(Lu)^{\frac{1}{2}}} \frac{\exp(-\lambda^2)}{\operatorname{erfc}(\lambda)} \frac{\operatorname{erfc}\left(\frac{1}{2(Lu\,Fo_x)^{\frac{1}{2}}}\right)}{\exp(-\lambda^2/Lu)} - \frac{\operatorname{erfc}\left(\frac{1}{2(Fo_x)^{\frac{1}{2}}}\right)}{\operatorname{erfc}\lambda}\right\}. \quad (33)$$

The equation (30) gives the following transcendental equation for the determination of λ:

$$\frac{k_{12}}{(a_{12})^{\frac{1}{2}}} \frac{T_v - T_s}{\operatorname{erf}[\lambda/(a_{12})^{\frac{1}{2}}]} \exp(-\lambda^2/a_{12}) + \frac{T_v}{\operatorname{erfc}(\lambda)} \exp(-\lambda^2)$$
$$= \pi^{\frac{1}{2}} \lambda\, Ko \left\{1 - T_v \frac{Lu\,Pn}{1-Lu}\left(\frac{1}{Lu} \frac{\exp(-\lambda^2)}{\operatorname{erfc}(\lambda)} \times \frac{\operatorname{erfc}[\lambda/(Lu)^{\frac{1}{2}}]}{\exp(-\lambda^2/Lu)} - 1\right)\right\}. \quad (34)$$

Using a computer, one can easily obtain λ from equation (34), which leads to obtaining the exact analytical solutions (31)–(33) for the analysis of the process of freezing in a humid porous body.

4. SOME ILLUSTRATIVE RESULTS

Using a minicomputer WANG 2200, some examples have been treated. In the Figs. 1–3, the upper lines show $T(Fo_x)$ and the lower ones: $-1-\theta(Fo_x)$.

The diagrams can be interpretated in two different ways:

(a) For a given time, the figures show temperature and moisture distributions in a halfspace (the surface $X = 0$ corresponds to $Fo_x \to \infty$). The right side of the figures corresponds to the frozen layer. From these figures it can be seen that the moisture is highest at the freezing front. Its increase in the zone near this front is in connection with the formation of a region where the moisture is lower than the initial one.

(b) The figures represent the temperature and moisture potentials versus time, for a fixed space position. From the beginning of the process the temperature continuously decreases while moisture at first decreases and after that increases until the moment when the freezing front reaches the point under consideration.

Figure 1 represents the influence of Kossovitch number at the position of the freezing front. The minimum and maximum of the moisture contents does not depend on Ko.

Figure 2 illustrates the negligible influence of Luikov number both on the position of the freezing front and on the temperature distribution. But Lu influences considerably the moisture distribution.

Fully analogous is the influence of the Posnov number (Fig. 3).

It may be concluded that Pn and Lu characterize the effect of migration of moisture towards the freezing front accompanied with a formation of a zone with a moisture-content lower with comparison to the initial one. The lowering of the moisture content in this zone is roughly proportonal to the values of Lu and Pn.

In the mathematical model there is no limit on the moisture content minimum. This is why the computer results are valid only when the minimum theoretical moisture content is higher than the physically acceptable one. In other conditions the phenomenon is much more complicated and corresponds to a periodic spatial distribution of the frozen structure.

That is the reason to expect that for high enough values of Pn and Lu the freezing will be associated to the formation of "ice lens" experimentally registered in [1–2].

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SOLUTION EXACTE DE LA CONGELATION D'UN DEMI-ESPACE POREUX HUMIDE

Résumé—Un nouveau problème de type Stephan est posé pour lequel on a obtenu des solutions exactes de distributions de température et d'humidité, aussi bien que de position du front de congélation évoluant dans un demi-espace poreux humide. La solution analytique est programmée en BASIC et quelques exemples sont présentés sous forme de courbes.

EINE EXAKTE LÖSUNG FÜR DEN GEFRIERVORGANG IN FEUCHTEN, PORÖSEN HALBKÖRPERN

Zusammenfassung—Es wird eine neue, dem Stephan-Problem ähnliche Aufgabe definiert. Exakte Lösungen werden angegeben für die Temperatur—und die Feuchtigkeitsverteilung sowie für die Lage der fortschreitenden Gefrierfront in einem feuchten, porösen Halbkörper. Die analytische Rechenmethode ist in BASIK programmiert. Für einige illustrative Beispiele sind Ergebnisse in Diagrammform angegeben.

ТОЧНОЕ РЕШЕНИЕ ДЛЯ ПРОЦЕССА ЗАМЕРЗАНИЯ ВЛАЖНОГО ПОРИСТОГО ПОЛУПРОСТРАНСТВА

Аннотация — Рассматривается новая задача типа Стефана. Получены точные решения для распределений температуры и влагосодержания, а также для положения движущегося фронта замерзания во влажном пористом полупространстве. Аналитическое решение запрограммированно на языке BASIK. Некоторые примеры представлены графически.